Now the left side of (2) equals $p(p-2)^2(p^3+4p^2+8p+16)+r(5p^3+32p-64)$.

Since $p=\sqrt{3q+\dfrac{(a-b)^2+(b-c)^2+(c-a)^2}{2}}\geqslant\sqrt{3}$, so $5p^3+32p-64\geqslant 0$, and (2) in fact holds. This completes the solution.

Solution 4 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

Editor′s note : I am taking the liberty of summarizing the author's solution because his solution path involved using partial derivatives that resulted in complicated systems of algebraic equations. These complicated systems were solved with the use of mathematica. The method the author used to solve this problem is valid, but it is also long and involved. Hence, summary.

The conditions in the statement often problem, that a, b and c are positive real numbers with $ab+bc+ca=1$, allowed the author to write c as $c=\dfrac{1-ab}{a+b}$, and to then consider the statement of the problem as a function in two variables x and y with $x=a$ and $y=b$. This gives:

$$f(x,y)=\frac{x^3(x+y)}{y^2+1}+\frac{y^3(x+y)}{x^2+1}+\frac{64(x+y)}{x^2+y^2+xy+1}+\frac{(1-xy)^3}{(x+y)^4}.$$

The problem then becomes one of finding the stationary points of $f(x,y)$, and for this he computed $f_x=\dfrac{\partial f}{\partial x}$ and $f_y=\dfrac{\partial f}{\partial y}$ and set each equal to zero This gives a system of two equations in two unknowns, $\begin{cases} f_x=0 \\ f_y=0 \end{cases}$.

Suppose that (a,b) satisfies the system of equations. (As the author noted there is often more than one solution to the system). The point $f(a,b)$ is now a candidate for being a stationary point; that is, for being a local max or a local min or a saddle point on the graph of $z=f(x,y)$. The usual classification criteria were used to classify the point (a,b).

- If $f_{xx}f_{yy}-f_{xy}^2<0$ at (a,b), then (a,b) is a saddle point.

- If $f_{xx}f_{yy}-f_{xy}^2>0$, at (a,b) then (a,b) is either a local maximum value or a local minimum value.

  We distinguish between them as follows:

- If $f_{xx}<0$ and $f_{yy}<0$ at (a,b), then (a,b) is a local maximum point.
- If $f_{xx}>0$ and $f_{yy}>0$ at (a,b), then (a,b) is a local minimum point.
- But if $f_{xx}f_{yy}=0$, at (a,b), then more advanced methods must be used to classify the solution point to the system of equations.

  With the help of mathematica it was shown that the equation $f(a,b)=34$ is its minimum.

Also solved by the proposer.

- **5573:** *Proposed by D.M.Bătinetu-Giurgiu, National College "Matei Basarab," Bucharest, Anastasios Kotronis, Athens, Greece, and Neulai Stanciu, "George Emil Palade" School,* *Buzău,*

Let $f:\Re_+\to\Re_+$ be a continuous function such that $\lim\limits_{x\to\infty}\dfrac{f(x)}{x^2}=a\in\Re_+$. ($\Re_+$ stands for the positive real numbers.) Calculate:

$$\lim_{x\to\infty}\left(\sqrt[n+1]{\prod_{k=1}^{n+1}\frac{f(k)}{k}}-\sqrt[n]{\prod_{k=1}^{n}\frac{f(k)}{k}}\right).$$

Solution 1 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

Let $a_n=\sqrt[n]{\prod\limits_{k=1}^{n}\frac{fk}{k}}$, and let $x_n=\left(\frac{a_n}{n}\right)$. Then, by Stolz-Cesaro:

$$\begin{aligned}\lim_{n\to+\infty}\ln\left(\frac{a_n}{n}\right) &= \lim_{n\to+\infty}\ln\left(\frac{n\ln a_n-n\ln n}{n}\right)=\lim_{n\to+\infty}\frac{\sum_{k=1}^{n}\ln\frac{f(k)}{k}-n\ln n}{n}\\ &= \lim_{n\to+\infty}\left[\ln\frac{f(n+1)}{(n+1)}-(n+1)\ln(n+1)+n\ln n\right]\\ &= \lim_{n\to+\infty}\left[\ln\frac{f(n+1)}{(n+1)^2}-n\ln\left(1+\frac{1}{n}\right)\right]=\ln\left(\frac{a}{e}\right).\end{aligned}$$

Hence, $\dfrac{a_n}{n}=\sqrt[n]{x_n}\dfrac{a}{e}$, which implies that $\dfrac{x_{n+1}}{x_n}\to\dfrac{a}{e}$. Moreover,

$$\frac{a_{n+1}}{a_n}\to 1,\text{ and }\left(\frac{a_{n+1}}{a_n}\right)^n=\frac{(n+1)^n x_{n+1}^{n/n+1}}{nx_n}=\left(1+\frac{1}{n}\right)^n\cdot\frac{x_{n+1}}{x_n}\cdot\frac{1}{\sqrt[n+1]{x_{n+1}}}\longrightarrow e.$$

Therefore,

$$\begin{aligned}\sqrt[n+1]{\prod_{k=1}^{n+1)}\frac{f(k)}{k}}-\sqrt[n]{\prod_{k=1}^{n}\frac{f(k)}{k}} &= a_{n+1}-a_n\\ &= \frac{a_n}{n}\cdot\frac{\left(\frac{a_{n+1}}{a_n}-1\right)}{\ln\left[1+\left(\frac{a_{n+1}}{a_n}-1\right)\right]}\ln\left[\left(\frac{a_{n+1}}{a_n}\right)^n\right]\longrightarrow\frac{a}{e}.\end{aligned}$$

Solution 2 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

By the Stolz-Cezaro lemma, the proposed limit, say L, is

$$L=\lim_{n\to\infty}\frac{\sqrt[n+1]{\prod\limits_{k=1}^{n+1}\frac{f(k)}{k}}}{n+1}=\lim_{n\to\infty}\sqrt[n+1]{\prod_{k=1}^{n+1}\frac{f(k)}{k(n+1)}}.$$

Therefore, by the root-quotient criterion,

$$L = \lim_{n\to\infty} \frac{\displaystyle\prod_{k=1}^{n+1} \frac{f(k)}{k(n+1)}}{\displaystyle\prod_{k=1}^{n} \frac{f(k)}{kn}} = \lim_{n\to\infty} \frac{n^n f(n+1)}{(n+1)^{n+1}(n+1)}.$$

Since $\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e}$, and $\lim_{n\to\infty} \frac{f(n+1)}{(n+1)^2} = a$, then $L = \frac{a}{e}$.

Solution 3 by Bib Pan, San Mateo, CA

First for any $\epsilon > 0$ there exists an N such that for all $k > N$

$$1 - \epsilon < \frac{f(k)}{ak^2} < 1 + \epsilon.$$

$$\begin{aligned} L &= \lim_{n\to+\infty} \left(\sqrt[n+1]{\prod_{k=1}^{n+1} \frac{f(k)}{k}} - \sqrt[n]{\prod_{k=1}^{n} \frac{f(k)}{k}} \right) \\ &= \lim_{n\to+\infty} \left(a \sqrt[n+1]{(n+1!)} \sqrt[n+1]{\prod_{k=1}^{n+1} \frac{f(k)}{k^2}} - a \sqrt[n]{n!} \sqrt[n]{\prod_{k=1}^{n} \frac{f(k)}{ak^2}} \right). \end{aligned}$$

From a known result (e.g., see "Problems in Real Analysis: Advanced Calculus on the Real Axis" by Radulescu T.L., et. al., P1.2.3)

$$\lim_{n\to+\infty} \left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} \right) = \frac{1}{e}.$$

It is easy to see,

$$a \left(\sqrt[n+1]{M(1-\epsilon)^{a+1-N}} \sqrt[n+1]{(n+1)!} - \sqrt[n]{M(1-\epsilon)^{n-N}} \sqrt[n]{n!} \right) \leqslant L$$

where

$$M = \prod_{k=1}^{N} \frac{f(k)}{ak^2}.$$

Letting $n \to +\infty$ on both sides, we have $L = \frac{a}{e}$.

Solution 4 by Arkady Alt, San Jose, CA

Let $a_n = \sqrt[n]{\prod_{k=1}^{n} \frac{f(k)}{k}}, n \in \mathbb{N}$ and $b_n := \frac{a_n^n}{n^n} = \frac{1}{n^n n!} \prod_{k=1}^{n} f(k)$.

First we will find $\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} \sqrt[n]{b_n}$. Since $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \left(\frac{n^n}{(n+1)^{n+1}} \cdot \frac{f(n+1)}{n+1} \right) =$

$\lim_{n\to\infty}\left(\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{f(n+1)}{(n+1)^2}\right)=\frac{a}{e}$ then by Geometric Mean Limit Theorem (Multiplicative Cesaro's Theorem) $\lim_{n\to\infty}\frac{a_n}{n}=\lim_{n\to\infty}\sqrt[n]{b_n}=\frac{a}{e}$. Coming back to the limit of the problem we obtain $L:=\lim_{n\to\infty}\left(\sqrt[n+1]{\prod_{k=1}^{n+1}\frac{f(k)}{k}}-\sqrt[n]{\prod_{k=1}^{n}\frac{f(k)}{k}}\right)=\lim_{n\to\infty}(a_{n+1}-a_n)=\lim_{n\to\infty}\frac{a_n}{n}\cdot n\left(\frac{a_{n+1}}{a_n}-1\right)=$ $\frac{a}{e}\lim_{n\to\infty}n\left(\frac{a_{n+1}}{a_n}-1\right)$. Since $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{a_{n+1}}{n+1}}{\frac{a_n}{n}}\cdot\frac{n+1}{n}=1$ implies $\lim_{n\to\infty}\ln\left(\frac{a_{n+1}}{a_n}\right)=0$ then $\lim_{n\to\infty}n\left(\frac{a_{n+1}}{a_n}-1\right)=\lim_{n\to\infty}\left(\frac{e^{\ln\frac{a_{n+1}}{a_n}}-1}{\ln\frac{a_{n+1}}{a_n}}\cdot n\ln\frac{a_{n+1}}{a_n}\right)=\lim_{n\to\infty}\left(n\ln\frac{a_{n+1}}{a_n}\right)=\ln\left(\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^n\right)=$
$\ln\left(\lim_{n\to\infty}\left(\frac{a_{n+1}^{n+1}}{a_n^n}\cdot\frac{n+1}{a_{n+1}}\cdot\frac{1}{n+1}\right)\right)=\ln\left(\lim_{n\to\infty}\left(\frac{f(n+1)}{n+1}\cdot\frac{n+1}{a_{n+1}}\cdot\frac{1}{n+1}\right)\right)=$
$\ln\left(\lim_{n\to\infty}\left(\frac{f(n+1)}{(n+1)^2}\cdot\frac{n+1}{a_{n+1}}\right)\right)=\ln\left(\lim_{n\to\infty}\frac{f(n+1)}{(n+1)^2}\cdot\lim_{n\to\infty}\frac{n+1}{a_{n+1}}\right)=\ln\left(a\cdot\frac{e}{a}\right)=1$
and, therefore, $L=\frac{a}{e}$

Also solved by Michel Bataille, Rouen, France; Brian Bradie, Christopher Newport University, Newport News, VA; Bruno Salgueiro Fanego, Viveiro, Spain; Moti Levi, Rehovot, Israel; Kee-Wai Lau, Hong Kong, China; Daniel Văcaru, Pitesti, Romania; Albert Stadler, Herrliberg, Switzerland; Romania;, and the proposers.

- **5574:** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu," Mehedinti, Romania*

Prove: If $0 < a \leqslant b \leqslant c$ then:

$$\frac{1}{1+e^{a-b+c}}+\frac{1}{1+e^b}\leqslant\frac{1}{1+e^a}+\frac{1}{1+e^c}.$$

Solution 1 by Titu Zvonaru, Comănesti, Romania

Clearing denominators, the given inequality is equivalent to:

$$(2+e^{a-b+c}+e^b)(1+e^a)(1+e^c)\geqslant(2+e^a+e^c)(1+e^{a-b+c})(1+e^b)$$

$$2+e^{a-b+c}+e^b+2e^a+e^{2a-b+c}+e^{a+b}+2e^c+e^{a-b+2c}+e^{b+c}+2e^{a+c}+e^{2a-b+2c}+e^{a+b+c}$$

$$\leqslant 2+e^a+e^c+2e^{a-b+c}+e^{2a-b+c}+e^{a-b+2c}+2e^b+e^{a+b}+e^{b+c}+2e^{a+c}+e^{2a+c}+e^{a+2c}$$

$$e^a+e^c+e^{2a-b+2c}+e^{a+b+c}\leqslant e^{a-b+c}+e^b+e^{2a+c}+e^{a+2c}$$

$$(e^b-e^a)(e^{a-b}-1)(e^{a+c}-1)\geqslant 0.$$

The equality holds if and only if $a=b$ or $b=c$

Solution 2 by Brian Bradie, Christopher Newport University, Newport News, VA